20.10

THE

MATHEMATICAL GAZETTE.

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ON SOME SEMI-REGULAR SOLIDS.

BESIDES the five regular solids, there are other polyhedra which satisfy some, but not all, the conditions of regularity as usually defined, and so may properly be termed semi-regular. Some of these, which have an interest both as geometrical figures and as naturally connected with the geometrical problems of the partition of space and of crystallography, I propose here briefly to discuss.

The problem of the partition of space into identically equal and similarly oriented cells, or of the complete filling of space with identically equal blocks with plane faces, is satisfied by cubes alone among the regular solids; for the cube is the only regular solid whose dihedral angle is an aliquot part of four right angles, and this is an essential condition that the space around a common edge may be exactly filled by the solids meeting in that edge. Another such partition, however, is possible into cells, each of which is a *rhombic dodecahedron* (or *twelve-face*), a solid whose construction and properties it is therefore of interest to examine.

The *Rhombic Dodecahedron* (or *Twelve-Face*).*

Conceive a cube surrounded by other cubes completely filling the space around it. Of these six have with it a common face, twelve others a common edge only, and eight a common corner only. Let AOA', BOB', COC', DOD' be the four diagonals of this cube, and let P, P' be the centres of the adjacent cubes on the faces $ABCD$, $A'B'C'D'$ respectively, Q, Q' the centres of those on $ABD'C'$, $A'B'DC$, and R, R' of those on $ADB'C'$, $A'D'BC$: then on each face as $ABCD$ we have a pyramid as $P(ABCD)$, whose four triangular faces make each with $ABCD$ an angle of half a right angle, and therefore two triangular faces as PAB, QAB which meet in a common edge AB are in the same plane and form together a rhombus $PAQB$. The twelve rhombuses through

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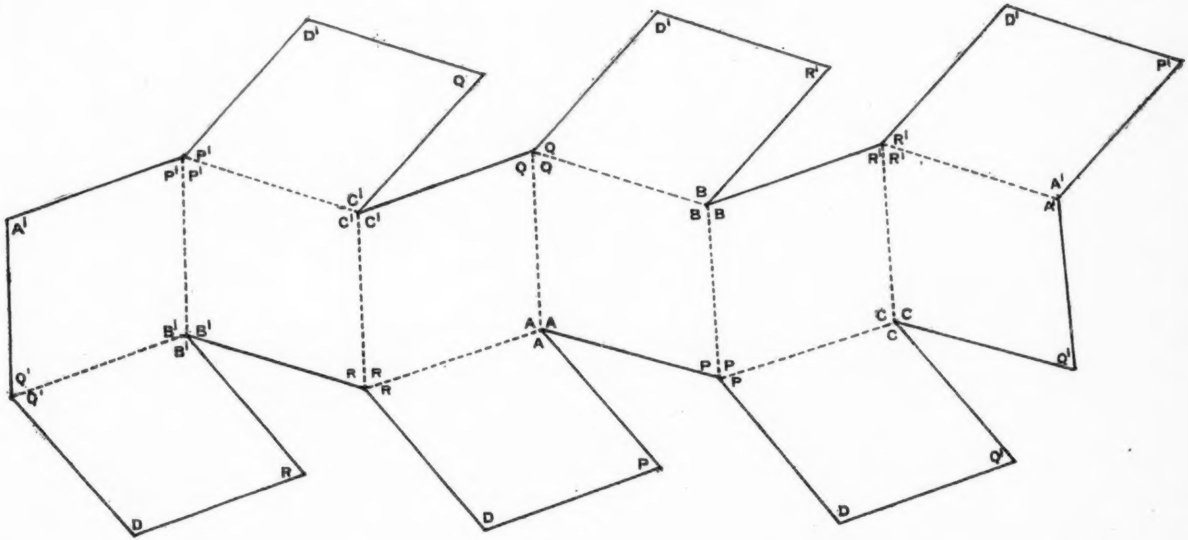
* I have found it often convenient to use the shorter English words *Twelve-Face*, *Fourteen-Face*, etc., instead of the polysyllabic Greek *Dodecahedron*, *Tetrakaidekahedron*, etc.



It is easy to construct a rhombic twelve-face by cutting the figure above out of a piece of cardboard, and half cutting through it along the dotted lines, so that the figure may be turned about each: then if folded over so as to bring together the points indicated by the same letters, the solid will be constituted, and a little sealing wax at the corners will make it fast. Constructions on the same principle for the other solids discussed in this paper are easily invented.

the twelve edges of the original cube thus form a solid having twelve identically equal rhombic faces, which is therefore called the Rhombic Dodecahedron (or Twelve-Face), while the cube may be called its cubic nucleus.

From this construction it is easily seen that the Rhombic Dodecahedron (or Twelve-Face) has twelve identically equal faces, which are rhombic but not rectangular, and therefore not *regular*; twenty-four equal edges; and fourteen corners, of which eight are the corners (A, B, etc.) of the original cube, and six at the centres (P, P', etc.) of the adjacent cubes. The eight corners are trihedral, since the solid angles at A, B, etc., are contained by the three planes passing through the three edges which meet at each point, while the six corners are tetrahedral, being the vertices of pyramids standing on the square faces of the cubic nucleus.

The rhombic faces are such that in each the two semi-diagonals and a side are in the ratio of $1 : \sqrt{2} : \sqrt{3}$. For in the rhombus $PAQB$, the diagonal AB is an edge of the cubic nucleus and the side PA is half a diagonal of the same, so that if the half of AB is taken as 1, PA is $\sqrt{3}$, and therefore half the other diagonal is $\sqrt{2}$.

Any trihedral angle of the twelve-face is contained by the three obtuse angles of its three rhombic faces, and any tetrahedral angle by the four acute angles of its four rhombic faces, and each edge connects a trihedral with a tetrahedral angle. Moreover, each tetrahedral angle, being one of the solid angles subtended at the centre of a cube by its six faces, is $\frac{1}{6}$ of the whole angular space about a point, or (if this is reckoned as made up of eight solid right angles) $\frac{4}{3}$ of a solid right angle; and each trihedral angle, being made up of a solid angle of a cube (*i.e.* a solid right angle) and three equal solid angles, such as $A(PBD)$ which is $\frac{1}{3}$ of a solid right angle, is equal to two solid right angles.

Also each dihedral angle between two adjacent faces, being the angle between two planes which pass through a diagonal of a cube and two of the edges which meet at one of its extremities, is $\frac{1}{3}$ of four right angles. For the three angles between the three planes OAB, OAC', OAD are obviously equal and together equal to four right angles.

Hence space may be divided into equal cells, each a rhombic twelve-face; any given cell having face-contact with twelve surrounding cells, each of its edges common to it and two others of these twelve cells, and each trihedral corner common to it and three others of the twelve, while each tetrahedral corner is common to it and 4 others of the twelve, and in addition one cell, which has neither face contact nor edge contact, but only point contact, with the given cell. In other words, while two cells meet in

each face, three meet in each edge, four meet at each trihedral corner, and six at each tetrahedral corner.

[For $3 \times \frac{1}{3}$ of 4 right angles = 4 right angles,
4×2 solid right angles = 8 solid right angles,
and $6 \times \frac{4}{3}$ solid right angles = 8 solid right angles.]

While the shorter diagonals of the rhombic faces form the edges of a cubic nucleus, the longer diagonals form those of a regular octohedronal nucleus, the trihedral corners being the vertices of pyramids on the triangular faces, and the tetrahedral corners the corners of the octohedron.

Spheres associated with the Twelve-Face.

Taking (as before) the edge of the cubic nucleus or the shorter diagonal of a face as 2, the distance of each four-faced (tetrahedral) corner from the centre O is 2 and that of each three-faced (trihedral) corner $\sqrt{3}$, so that no single sphere can be described about the twelve-face, but the four-faced corners lie on one sphere and the three-faced on a smaller concentric sphere. But the distace of each face from the centre, being the distance of the middle point of each edge of the cubic, or of the octohedric, nucleus from the centre, is $\sqrt{2}$, so that a sphere, whose radius is $\sqrt{2}$, can be inscribed in the twelve-face, and this is the sphere which touches each edge both of the cubic and of the octohedric nucleus at its middle point.

If a sphere be thus inscribed in each of the rhombic twelve-face cells which surround a given cell, the sphere inscribed in the given cell will be in contact with twelve other surrounding spheres at the intersections of the diagonals of the several faces of its cell, these faces being in fact common tangent planes to it and the adjacent spheres.*

It is plain that this arrangement of spheres is the most compact that can be made, and is therefore the arrangement which spherical shot or balls would assume, when shaken together in a box, or when spherical balls are arranged in a pile.

In a pile on a square (or rectangular) base each ball is in contact with four balls in the same horizontal layer, and with four balls each in the layer below and in that above it, and in this case the edges of the cubic nucleus of the cell are four of them vertical and the rest horizontal, or a diagonal of the twelve-face joining two four-faced corners is vertical. Also four of the cell faces are vertical planes. In a pile on a triangular base on the other hand, each ball is in contact with six balls in the same horizontal layer, and with three balls each in the layers below and above. In this case a diagonal of the cubic nucleus, which is also a diagonal of the twelve-face

* Cf. an interesting paper by Mrs. Bryant, D.Sc. (*Proc. Lon. Math. Soc.* xvi. No. 252), "On the Ideal Geometrical Form of Natural Cell-Structure."

joining opposite three-faced corners, is vertical, and the points of contact in the horizontal layer are the middle points of the six edges of the cube which do not meet that diagonal. Also six of the cell faces are vertical planes.

In this last case, if we suppose the vertical faces uniformly elongated in the vertical direction only, the other faces remaining the same, we shall have an hexagonal prism closed at each end by three rhombic faces. This is the form of the cells of the honey-bee, which, as is well-known and easily proved, for a given quantity of honey requires a minimum quantity of wax for its walls.

It is easily seen that the volume of the twelve-face cell is double of that of the cubic nucleus and is therefore sixteen cubic units, while that of the inscribed sphere is $\frac{4}{3}\pi(\sqrt{2})^3$ or $\frac{8\sqrt{2}}{3}\cdot\pi$, so that the ratio of the latter to the former is $\pi : 3\sqrt{2}$. Hence in the closest packing of a large number of spherical balls the space occupied by the balls themselves is to the total space required as $\pi : 3\sqrt{2}$ or rather less than $3:4$.

The conjugate Fourteen-Face (*or Tetrakaidekahedron*).

The lines which join the centre of each rhombic face to the centres of its four adjacent faces form the twenty-four edges of another solid, which has fourteen faces, eight of them equilateral triangles opposite the three-faced corners of the twelve-face, and six squares opposite the four-faced corners, and twelve corners, all of them four-faced. It may be constructed from the cubic nucleus by truncating each of the eight corners so as to leave in each of its six faces the square inscribed therein. This solid then is semi-regular in that its edges and solid angles are all equal, and its faces regular figures, but six of them are squares and eight triangles. It has obviously a circumscribing sphere, and there are two inscribed spheres, one touching all the triangular, the other all the square, faces. Blocks of this form, however, could not fill solid space without vacant interstices, since their dihedral angles, being each equal to the supplement of the angle between a diagonal and an edge of a cube, are not aliquot parts of four right angles.

There is, however, a semi-regular fourteen-face which does possess this property, and which Lord Kelvin ("On Homogeneous Division of Space," *Proc. Roy. Soc.*, lv. p. 331, Jan. 1894), has named

The Orthic Tetrakaidekahedron, or *Kelvin Fourteen-Face.*

This solid may be constructed from a regular octohedron, by dividing each edge into three equal parts and truncating the octohedron by planes opposite each of its six corners, passing through the four adjacent points of trisection. This will give a solid with six square faces opposite the corners, or perpendicular

to the three diagonals of the octohedron, and eight regular hexagonal faces in the planes of its faces. It will easily be seen that this solid has thirty-six equal edges, and twenty-four three-faced corners, each solid angle being contained by two hexagonal, and one square, face. Of the edges twenty-four are the intersections of a hexagonal and a square face, and twelve of two hexagonal faces. The dihedral angle between two hexagonal faces is the dihedral angle of a regular octohedron, and that between an hexagonal and a square face is the supplement of half the same, so that two of these latter and one of the former make up four right angles, and thus three "Kelvin fourteen-faces" may meet in a common edge, so as exactly to fill up the space about that edge.

It will have been observed that all the points, lines, and planes which occur in connection with the solids discussed above, have reference to (A) three rectangular directions in space, or the directions of the edges of a cube, (B) six directions bisecting the angles between each pair of the first three directions, or the six directions of the face diagonals of the cube, and (C) four directions equally inclined to the first three, or the four directions of the diagonals of the cube.

Thus the rhombic twelve-face cells in the partition of space have their four-face corners on lines parallel to the (A) directions, and their three-face corners on lines parallel to the (C) directions, while their faces are normal to the (B) directions, and the points of contact of the inscribed spheres are on lines in the same directions, and their edges are parallel to the (C) directions.

The general problem, of which it has only been possible here to touch the fringe, has been discussed by Bravais in the *Journal de l'Ecole Polytechnique, tome* 19, 1850; and by Lord Kelvin in his *Mathematical and Physical Papers*, vol. 3, as well as in his paper referred to above. Also by Sohncke, *Punksysteme als Grundlage einer Theorie der Krystallstructur*; Karlsruhe, 1876.

R. B. HAYWARD.

NOTES ON ELEMENTARY DYNAMICS. III.

COLLISION.

WE have seen that for solving problems on the simultaneous impact of more than two bodies, we require in addition to the laws of Abstract Dynamics, not only a *generalization* of Newton's Empirical Law as to change of relative velocity, but also the assumption that the impacts are simultaneous in the special sense of having their greatest compression at the same instant. In other words, it is only in a particular set of special cases that elementary methods can lead to a solution.

This seems to have been overlooked by many examiners in setting questions of this sort without the necessary restrictions.

I now proceed to touch on a few points regarding the theory of impacts which may be of interest to some who have learned the subject only from the elementary text books.

The use of such terms as "coefficient of elasticity," and "perfectly elastic body" in the theory of impact is perhaps to blame for the confusion that exists in the minds of many learners as to the true nature of the theory of collision. There is a vague impression that these terms imply a close connection between the Theory of Collision and that of Elastic Bodies. To prevent this confusion Thomson and Tait substituted the term "coefficient of restitution" for "coefficient of elasticity." Lord Kelvin has done much to give precision to our ideas regarding elasticity, and has summarized his views in the article on that subject in the *Encyclopædia Britannica.* Following him we may note the distinction between *statical* Perfect Elasticity, which implies that the forces required to keep a body at rest in a given state of *strain* or deformation are independent of the states of strain through which the body may have passed in reaching the state in question; and *kinetic* Perfect Elasticity, which implies that the forces are also independent of the rate at which the strain of the body may be changing, when not at rest. The former kind of Perfect Elasticity is possessed by many bodies, within certain limits, the latter probably by none, though for comparatively slow motions some bodies possess it approximately, within certain limits of strain. Lord Kelvin found by experiment that in wires subjected to torsional vibrations well within the limits of perfect statical elasticity, there is still a loss of energy due to a kind of internal resistance which he calls "viscosity" (as depending on the strain-velocity). This showed that the kinetic elasticity of such bodies is far from being perfect; and there is reason to believe that the same is true of all terrestrial substances.

The so-called "Perfect Elasticity" of bodies in the elementary theory of Impact is not to be identified with either of the above-defined qualities of bodies. It is attributed to bodies in which the "coefficient of restitution" is equal to 1, and so far as it can be related to the Theory of Elastic Bodies, is really a much more complex quantity than either. This can be best explained by considering more closely the physical nature of a collision.

When two or more bodies strike one another, there is in general a state of vibration set up in each body,* while at the

* This is perceived by the sense of touch when a batsman strikes a cricket ball, and by the ear when a bell is tolled.

same time the translational and rotational velocities of each body as a whole are changed. Broadly speaking, we may say that (1) part of the energy of the bodies concerned is turned into molecular vibrations, *i.e.* into heat, at the instant of collision, (2) part appears as energy of vibration of the bodies, passing continually from the kinetic to potential forms, and *vice versa* (as in the case of bells and other musical percussion-instruments), and (3) the remainder appears as kinetic energy in the motions of the bodies as wholes.

Part 2 soon gets dissipated, partly in setting up sound vibrations in the air, and partly into heat, in virtue of the "viscosity" or want of perfect elasticity of the substances, but not as a rule before the collision is over. Part 3 only is dealt with in the elementary theory of Impact.

It will be seen that even on the supposition of perfect static and kinetic elasticity in the bodies considered, it does not follow that the coefficient of restitution would be unity. The problem of calculating the exact nature of the motion after impact, even with such an ideal assumption, is a difficult one, and even for the simple case of the direct impact of two equal homogeneous spheres, the solution cannot be given in finite terms. One case only has been solved in finite terms, viz., that of two uniform cylinders colliding directly in the line of their common axis. There it is found that if the cylinders be equal, they will *exchange* velocities, and separate without vibration, while if unequal, the shorter will have no vibration after impact, but the longer will be set in vibration longitudinally. The result when the cylinders are equal corresponds so far with the elementary theory of the impact of bodies whose coefficient of restitution is unity, but it has been found by experiment that the time of duration of impact is much greater than that given by the above calculation; and it is probable that the assumption of "perfect kinetic elasticity" for the rapid strain-changes during impact is quite erroneous.

Various other attempts have been made, with more or less success, to explain Newton's Empirical Law by the abstract Theory of Elastic Bodies with the aid of certain assumptions. These have been well explained by A. E. H. Love in his *Treatise on Elasticity*, Vol. II. Enough has been said, however, to show that the Elementary Theory of Impact stands on an empirical basis of its own, and that the terms "coefficient of elasticity" and "perfect elasticity" would be better omitted entirely from its nomenclature.

It may be added that the term "coefficient of elasticity," or, briefly, "elasticity," is applied by writers on the theory to certain quantities connecting the components of stress with those of strain in a given substance. The term "Modulus of

Elasticity" is defined by Thomson and Tait as "the number obtained by dividing the number expressing a stress by the number expressing the strain it produces." It is sometimes inaccurately called a "coefficient of elasticity." The term "resilience" is sometimes used as a substitute for "coefficient of restitution" in elementary Impact-theory: but though not so misleading as the term "coefficient of elasticity," is perhaps better reserved for other uses. See "Elasticity," §§ 53-66, in the *Encyc. Brit.*

R. F. MUIRHEAD.

MATHEMATICAL NOTES.

32. *On the proof of the formula* $S=ut+\frac{1}{2}ft^2$.

I should like to elicit opinions from mathematical teachers as to how far ordinary students can be expected to grasp what is implied in the proofs of this formula which dispense with the notion of infinitesimals, by introducing instead the conception of 'mean velocity.' It seems to me that, properly understood, these proofs imply all that is explicitly stated in such a proof as that given in Todhunter's elementary book; and that if the new proof appears simpler to the student, it is only because he does not understand it—because to him it is merely "a fudge." In the *Elements of Dynamics*, by the Rev. J. L. Robinson, for example, the proof commences with the statement (p. 47):

"Since the velocity increases *uniformly* throughout the given time, the *mean* velocity during the interval will be *half the sum of the extreme velocities.*"

I am sure most students would accept this statement as a mere truism. Very possibly they would even fail to reproduce it if asked to write out the proof, and merely say:

$$\begin{aligned} &\text{Velocity at beginning} &&= u,\\ &\quad\;\; ,, \quad\;\; \text{end} &&= u+at,\\ \therefore\;&\text{Mean velocity} &&= u+\tfrac{1}{2}at,\\ \therefore\;&\text{Space described} &&= ut+\tfrac{1}{2}at^2.\end{aligned}$$

And if they were asked to find the space described under uniformly increasing acceleration they would cheerfully proceed to do so, thus:

$$\begin{aligned} &\text{Acceleration at beginning} &&= a,\\ &\qquad\quad ,, \qquad \text{end} &&= a+\beta t,\\ \therefore\;&\text{Mean acceleration} &&= a+\tfrac{1}{2}\beta t,\\ \therefore\;&\text{Velocity at time } t &&= u+at+\tfrac{1}{2}\beta t^2.\end{aligned}$$

But velocity at beginning $=u$,

$$\begin{aligned} \therefore\;&\text{Mean velocity} &&= u+\tfrac{1}{2}at+\tfrac{1}{4}\beta t^2,\\ \therefore\;&\text{Space described} &&= ut+\tfrac{1}{2}at^2+\tfrac{1}{4}\beta t^3.\end{aligned}$$

It is true that in another part of the book in question a careful definition of 'mean velocity' is given; but the apparent simplicity of the proof is due to the fact that this definition is forgotten, or its force ignored. To really convince oneself that the 'mean velocity' is the arithmetic mean of the extreme velocities *because* the acceleration is uniform, one has to go through a process of reasoning not less complex than that given by Todhunter.

Prof. S. L. Loney in his *Elementary Dynamics* attempts—but not I think quite successfully—to exhibit the reasoning more clearly. He says (p. 28):

"Now the velocity increases *uniformly* throughout the interval t. Hence the velocity at any instant preceding the middle of this interval by time T is as much less than V as the velocity at a moment at the same time T after the middle of the interval is greater than V."

"Hence, since the time t could be divided into pairs of such corresponding instants, the space described is the same as if the point moved for time t with uniform velocity V."

Here there seems to be a confusion between an instant or moment of time *at* which the point has a particular velocity and a small interval *during* which it describes a small space, and the reasoning by which it may be shown that we may in the limit treat the velocity as uniform during such a small interval is suppressed. In fact, when properly explained, the argument again would turn out to be no more simple than Todhunter's. No doubt the teacher can see the force of the reasoning without going into details; but if the student accepts it readily, it is almost certainly because he fails to see all that is implied in the specious phraseology.

I am glad to see that both Messrs. Robinson and Loney give the old proof as an alternative. But if the student has to learn both proofs, it cannot be urged that the new proof is introduced to save him trouble, any more than, I hope, it is presented as a model of formal reasoning.

EDWARD T. DIXON.

33. *Note on Division.*

If $\mathrm{f}(x)$ be any rational integral function of x, we know that

$$\frac{\mathrm{f}(x)}{x-a}=\text{a quotient } Q+\frac{\mathrm{f}(a)}{x-a},$$

$$\therefore \quad Q=\frac{\mathrm{f}(x)-\mathrm{f}(a)}{x-a}.$$

This may be used to write down the value of Q. Thus, if

$$\mathrm{f}(x)=ax^3+bx^2+cx+d,$$

then

$$Q=a\frac{x^3-a^3}{x-a}+b\frac{x^2-a^2}{x-a}+c\frac{x-a}{x-a}$$

$$=a(x^2+ax+a^2)+b(x+a)+c.$$

If

$$\mathrm{f}(x)=(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n),$$

$$=(\overline{x-a}+\overline{a-a_1})(x-a_2)\ldots\ldots(x-a_n),$$

then

$$\mathrm{f}(a)=(a-a_1)(a-a_2)\ldots\ldots\ldots\ldots\ldots(a-a_n),$$

$$\therefore \quad \frac{\mathrm{f}(x)-\mathrm{f}(a)}{x-a}=(x-a_2)(x-a_3)\ldots(x-a_n)$$

$$+(a-a_1)\frac{(x-a_2)\ldots(x-a_n)-(a-a_2)\ldots(a-a_n)}{x-a}.$$

We may now put $x-a_2=\overline{x-a}+\overline{a-a_2}$ and continue the process. The quotient then may be written down in the useful form

$$(x-a_2)(x-a_3)\ldots(x-a_n)+(a-a_1)(x-a_3)\ldots(x-a_n)$$
$$+(a-a_1)(a-a_2)(x-a_4)\ldots(x-a_n)+(a-a_1)(a-a_2)(a-a_3)(x-a_5)\ldots(x-a_n)$$
$$+\text{etc.} \ +(a-a_1)(a-a_2)\ldots(a-a_{n-1}),$$

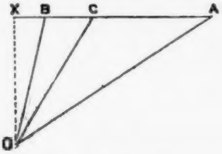
in which the a's gradually replace the x's, just as in $\dfrac{x^n-a^n}{x-a}$.

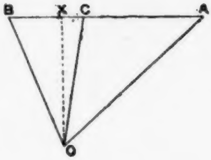
It is plain that on putting $a=x$ we obtain the usual expression for $\mathrm{f}'(x)$.

R. W. GENESE.

34. *Proof of the theorem that the sum of the moments of two forces which intersect is equal to the moment of their resultant.*

Let OA, OB be the lines along which the forces act, and X the point about which moments are taken; draw XAB perpendicular to OX. Also let λOA and μOB be the two forces; then it is known that their resultant is $(\lambda+\mu)OC$, where C is a point in AB such that $\lambda CA=\mu BC$.





Hence, paying proper attention to sign,

$$\lambda(XA-XC)=\mu(XC-XB),$$
$$\therefore\quad (\lambda+\mu)XC \quad =\lambda XA+\mu XB,$$
$$\therefore\quad (\lambda+\mu)OX\,.\,XC=\lambda OX\,.\,XA+\mu OX\,.\,XB,$$
$$\therefore\quad (\lambda+\mu)2\Delta OXC=\lambda 2\Delta OXA+\mu 2\Delta OXB.$$

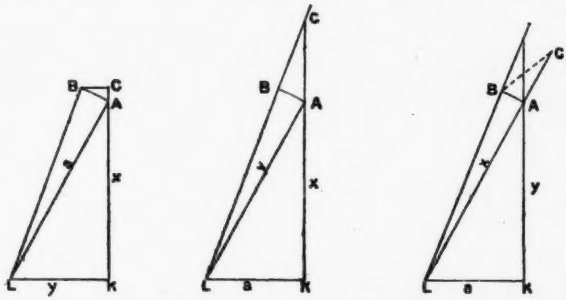
Therefore the algebraic sum of the moments of the two forces about X is equal to the moment of their resultant. E. M. LANGLEY.

35. *Method of finding the differential coefficients of the inverse circular functions by one rule.*

(i.) Let $\theta=\sin^{-1}\frac{x}{a}$ or $\tan^{-1}\frac{x}{a}$ or $\sec^{-1}\frac{x}{a}$.

Construct three right-angled triangles ALK, and let the angle $ALK=\theta$. Name the sides of these triangles x, a, y as in the figures, and the hypothenuse h (so that h will be one of the quantities x, a, y).

The value of y is afterwards to be expressed in terms of x and a.

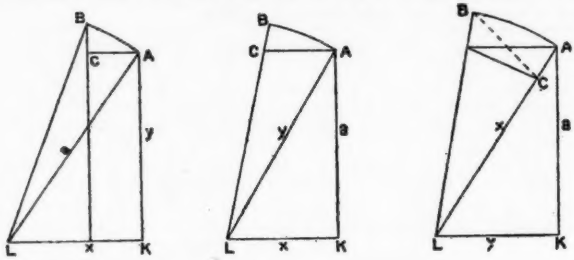


With centre L and radius LA describe a small arc AB and join LB. Let the angle ALB be the increase in θ. Complete the figures as shown, AC being the increase in x. The triangle ABC formed by the arc AB, the difference of x and the difference of y, will, in the limit, be similar to the triangle ALK.

In each case, $$\frac{d\theta}{dx}=\text{Lt.}\frac{AB}{AL}\cdot\frac{1}{AC}=\frac{1}{AL}\text{Lt.}\frac{AB}{AC}=\frac{1}{h}\cdot\frac{a}{y}.$$

(ii.) Let $\theta=\cos^{-1}\frac{x}{a}$ or $\cot^{-1}\frac{x}{a}$ or $\operatorname{cosec}^{-1}\frac{x}{a}$.

The same methods of construction as before will give the following figures, AC being again the difference of x, but being in each case a decrease.



In each of these cases

$$-\frac{d\theta}{dx}=\text{Lt.}\frac{AB}{AL}\cdot\frac{1}{AC}=\frac{1}{AL}\text{Lt.}\frac{AB}{AC}=\frac{1}{h}\cdot\frac{a}{y}.$$

So that in all cases,
$$\frac{d\theta}{dx}=\pm\frac{1}{h}\cdot\frac{a}{y}.$$

Example.—Let $\theta=\tan^{-1}\frac{x}{a}$, then $h=\sqrt{x^2+a^2}$ and $y=\sqrt{x^2+a^2}$.

$$\therefore \quad \frac{d\tan^{-1}\frac{x}{a}}{dx}=\frac{1}{\sqrt{x^2+a^2}}\cdot\frac{a}{\sqrt{x^2+a^2}}=\frac{a}{x^2+a^2}.$$

The sign of the result is easily obtained by inspection.

Method of obtaining the differential coefficients of the circular functions by one rule.

Let ALK be a right-angled triangle and let the angle ALK be called θ. Let p stand for the side opposite θ, h for the hypotenuse and b for the remaining side. Then each circular function of θ can be written as a ratio, making use of two of the symbols p, h, b. Call the symbol not used in the ratio the 3rd symbol.

Then the differential coefficient of each circular function of θ is equal to

$$\pm\frac{h\,.\,\text{3rd symbol}}{(\text{denominator of the ratio})^2}.$$

The proof of this rule can be obtained from the figures given for the previous rule by letting $\frac{x}{a}=\sin\theta$, $\tan\theta$, $\sec\theta$, etc., in order.

Example.—Find $\frac{d\sec\theta}{d\theta}$. Since $\sec\theta=\frac{h}{b}$, the 3rd symbol is p

$$\therefore \quad \frac{d\sec\theta}{d\theta}=\frac{hp}{b^2}=\sec\theta\tan\theta.$$

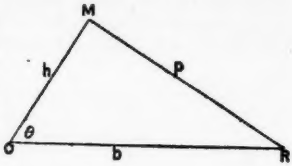
The sign of the result must be obtained by inspection as before.

The rules given above for circular functions may be further adapted to the hyperbolic functions as follows :

Let θ be the angle whose circular functions are so connected with the hyperbolic functions of u that $\cosh u = \sec\theta$, and $\sinh u = \tan\theta$.

Then θ is called the Gudermannian of u.

Construct a right-angled triangle OMR with the right angle at M, and let the angle $MOR = \theta =$ Gudermannian of u. Let the side opposite θ be called p, and the other two sides h and b as marked.



Then $$\sinh u = \frac{p}{h}, \quad \cosh u = \frac{b}{h}, \quad \tanh u = \frac{p}{b}, \text{ etc.}$$

Let $u = \sinh^{-1}\frac{x}{a}$, $\cosh^{-1}\frac{x}{a}$, $\tanh^{-1}\frac{x}{a}$, etc., in order, then x, a, y can represent the three sides of the triangle, y being later expressed in terms of x and a by Euc. I. 47.

Then in all cases $$\frac{du}{dx} = \pm\frac{1}{h}\cdot\frac{a}{y},$$

where h stands always for OM. The sign must be determined by inspection.

Examples.—(i.) Let $u = \sinh^{-1}\frac{x}{a}$; then $p = x$, $h = a$, $y = \sqrt{a^2+x^2}$;

$$\therefore \quad \frac{du}{dx} = \frac{1}{h}\cdot\frac{a}{y} = \frac{1}{a}\cdot\frac{a}{\sqrt{a^2+x^2}} = \frac{1}{\sqrt{a^2+x^2}}.$$

(ii.) Let $u = \tanh^{-1}\frac{x}{a}$; then $p = x$, $b = a$, $y = \sqrt{a^2-x^2} = h$;

$$\therefore \quad \frac{du}{dx} = \frac{1}{h}\cdot\frac{a}{y} = \frac{1}{\sqrt{a^2-x^2}}\cdot\frac{a}{\sqrt{a^2-x^2}} = \frac{a}{a^2-x^2}.$$

The rule for the hyperbolic functions is the same as for the circular functions if the triangle OMR, with the symbols as marked, be used. Thus each hyperbolic function of u can be written as a ratio by making use of two of the symbols. Call the symbol not used the 3rd symbol.

$$\text{Differential coefficient required} = \pm\frac{h\,.\,\text{3rd symbol}}{(\text{denominator of the ratio})^2}.$$

Example.—Find the differential coefficient of $\tanh u$.

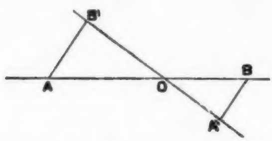
$$\tanh u = \frac{p}{b}, \quad \therefore \text{ req. diff. coeff.} = \frac{h\,.\,h}{b^2} = \operatorname{sech}^2 u.$$

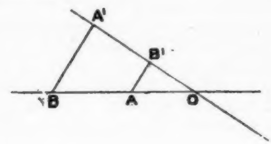
The proper positions for the symbols p, h, b can be easily remembered by noticing that both in the circular and hyperbolic functions h is applied to the side whose rotation increases or decreases θ, and p to the side opposite θ, this side being perpendicular in the triangle OMR, when turned into its right position as an ordinate of the rectangular hyperbola. This is very clearly seen from a figure showing the connection between θ and u in Professor Greenhill's *Differential and Integral Calculus.*

F. WHATLEY.

36. *On Involution Ranges.*

(i.) The following is a simple geometrical proof of the theorem that pairs of points in a line, called conjugates of each other, AA', BB', CC', DD', etc., are in involution* if $OA \,.\, OA' = OB \,.\, OB' =$ etc. $=$ constant, O being some fixed point in the line, called the centre of the involution.





Set off OA', OB' on any line making an angle with OA.

Then, since $\dfrac{OA}{OB'} = \dfrac{OB}{OA'}$, AB' is parallel to $A'B$.

$$\therefore \quad \frac{AB}{OA} = -\frac{A'B'}{OB'}, \quad \frac{CD}{OC} = -\frac{C'D'}{OD'}, \quad \frac{OA}{AD} = -\frac{OD'}{A'D'}, \quad \text{and} \quad \frac{OC}{CB} = -\frac{OB'}{C'B'}.$$

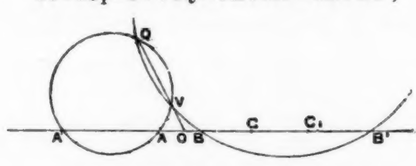
Compounding, we have $$\frac{AB \,.\, CD}{AD \,.\, CB} = \frac{A'B' \,.\, C'D'}{A'D' \,.\, C'B'},$$

$$\therefore \quad (ABCD) = (A'B'C'D'),$$

using the ordinary notation for cross ratios. Thus the cross ratio of any four points is equal to that of their conjugates.

(ii.) The converse is also easily proved.† Let the cross ratio of any four points whatever of the pairs AA', BB', etc., be equal to that of their conjugates. Take any point V not in AB, and draw the circles $AA'V$, $BB'V$ to meet in Q; draw QV to O, and determine C_1 so that

$$OC \,.\, OC_1 = OV \,.\, OQ = OA \,.\, OA' = OB \,.\, OB' ;$$



then by (i.), $\qquad (AA'BC) = (A'AB'C_1)$;

but by hypothesis $\qquad (AA'BC) = (A'AB'C')$,

since the cross ratio of any four is equal to that of their conjugates ; hence

$$(A'AB'C_1) = (A'AB'C'),$$

and therefore C_1 coincides with C' ;

$$\therefore \quad OC \,.\, OC' = OA \,.\, OA' = OD \,.\, OD', \text{ etc.}$$

E. Budden.

37. *General proof of the pole and polar property of the circle*, or any curve of the second order having the cross ratio property of the circle. Such curves are called below cross-ratio curves.

For proof of Pascal's theorem assumed below, see Richardson and Ramsay's *Mod. Pl. Geometry*, viii. 17 ; Mulcahy's *Mod. Geometry*, § 27 ; or Taylor's *Euclid* (Pitt Press), vi. Ad. Pr. 21.

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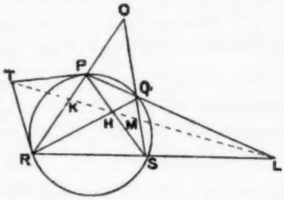
*Pairs of points AA', BB', ... on the same straight line are said to be in involution when the cross ratio of any four points whatever is equal to that of their conjugates.

†This form of proof is adapted from Mulcahy's *Modern Geometry*, § 35.

Theorem and definition. The locus of the harmonic conjugate of a fixed point with respect to the two points in which any straight line through the point cuts a cross-ratio curve is a straight line, and is called the polar of the point.



Let O be the point; take a fixed chord OPR, divide it harmonically at K so that $(OPKR)=-1$, draw the tangents PT, RT to meet in T; then TK is a fixed line.

Then if $OQMS$ be any other chord divided harmonically at M, we shall show that the locus of M is the straight line TK.

Join PS, QR to meet in H, and PQ, RS to meet in L. Then the two coincident points P, the point Q, the two coincident points R, and the point S, form the six vertices of a Pascal hexagon $PPQRRS$.

Therefore the intersections of opposite sides

$$\left.\begin{array}{l} PP,\ RR,\ i.e.\ T \\ PQ,\ RS,\ i.e.\ L \\ QR,\ SP,\ i.e.\ H \end{array}\right\}\text{ are collinear (Pascal's Theorem).}$$

Hence LH passes through T; but LH cuts OPR and OQS harmonically, and therefore passes through K and M (harmonic properties of quadrilaterals).

Hence M is on the fixed line TK, *i.e.* the locus of M is a straight line.

Corollary.—Since OPR was chosen arbitrarily, and the tangents PT, RT intersect on the polar of O, it will follow that the tangents at the extremities of any chord OQS intersect on the polar of O. Thus the polar is also the locus of intersections of tangents at the extremities of chords through the fixed point.

E. Budden.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the Gazette *attractive by sending either original or selected problems.*

Solutions of problems should be sent within three months of the date of publication. They should be written only on one side of the paper, and without the use of contractions not intended for printing. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

151. Prove that $\sqrt{9\sqrt{6}+6\sqrt{12}}+\sqrt{9\sqrt{6}-6\sqrt{12}}=2\sqrt[4]{216}$.

152. If $z=\cdot 9999999999$, and $e=2\cdot 71828$; find the value of $z+\frac{1}{2}z^2+\frac{1}{3}z^3+\ldots$ to four decimals.

153. A segment of a circle is placed with its base vertical; prove that the time from rest down any chord through the highest point is constant if the coefficient of friction has a certain fixed value.

154. Prove that any weight up to $\frac{1}{2}(3^n-1)$ ounces can be determined correct to an ounce by using a pair of scales and n fixed weights.

155. Construct the circle which passes through a given point and cuts two given circles orthogonally.

156. The triangle of minimum perimeter circumscribed to a given acute-angled triangle is the triangle whose in-circle is the circum-circle of the given triangle.

157. Prove that the six planes perpendicular to the edges of a tetrahedron through their middle points meet in a point.

158. If squares be described outwards on the sides of a convex quadrilateral the line joining the centres of two opposite squares is equal to the line joining the centres of the other two.

159. Prove that all triangles circumscribed to a rectangular hyperbola are obtuse-angled.

160. If $a+b+c+d+e+f=0$,

and $a^3+b^3+c^3+d^3+e^3+f^3=0$,

prove that

$$(a+c)(a+d)(a+e)(a+f)=(b+c)(b+d)(b+e)(b+f).$$

(Cambridge.)

161. Find the H.C.F. of

$$(x^2-a^2-b^2+c^2)^2+4(ax-bc)^2 \text{ and } (x-b)^4-(a-c)^4.$$

(Cambridge.)

162. Prove that

$$\sin^3\alpha\sin(\beta-\gamma)+\sin^3\beta\sin(\gamma-\alpha)+\sin^3\gamma\sin(\alpha-\beta)$$
$$=-\sin(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

(Cambridge.)

163. Sum to n terms

$$1^2\cos\alpha-2^2\cos 2\alpha+3^2\cos 3\alpha-4^2\cos 4\alpha+\dots$$ (Cambridge.)

164. If Pp, Qq are normal chords of a conic at right angles to one another, the quadrilateral $PQpq$ has two sides parallel.

(Cambridge.)

165. Find the maximum and minimum values of

$$(x-1)^{\frac{4}{3}}(x-2)^{\frac{2}{3}}(x-3)^{\frac{1}{3}}.$$ (Cambridge.)

166. Four equal particles which repel one another with forces varying as the distance are fastened to an endless string. Prove that in the position of equilibrium the line joining two of the particles is parallel to the line joining the other two.

(Cambridge.)

167. Three equal balls (coefficient of elasticity e) are at rest in a straight line on a smooth horizontal plane. If A be projected towards B, show that there will be at least five impacts if

$$e+e^{-1}>4+2\sqrt{5}.$$ (Cambridge.)

168. The points on a given line whose distances from two given points have a maximum or minimum ratio are the ends of a diameter of a circle through the given points.

169. Prove that the equation

$$\sqrt{a+x} \pm \sqrt{b+x} \pm \sqrt{c+x} \pm \sqrt{d+x} = 0$$

has only one root, and find it.

170. The sum of the reciprocals of all positive primes is infinite.

171. If any number of consecutive integers from 2 onwards be expressed in prime factors, the sum of the indices of the factors will be more often odd than even.

172. Circles are drawn through two fixed points. Trace the path of a point which cuts them all at the same angle; and find its equation.

173. Between two sides of a triangle show how to inflect a straight line which shall have given ratios to the segments of the sides between it and the base.

(Proposed in 1864 by Mr. R. TUCKER.)

174. If AP, IK be perpendiculars from the vertex A and in-centre I to the base BC of the triangle ABC, and the tangent to the in-circle parallel to BC meet AP in Q, and the other tangent from Q meet BC in V, prove that $VB \,.\, VC = VP \,.\, VK$.

E. BUDDEN.

175. When a conic S is reciprocated with respect to any point O into a conic S', prove that the reciprocals of the four foci of S are the two pairs of common chords of S' and a point-circle at O. R. P. ROYSTON.

176. If P is a point from which four normals are drawn to a given ellipse, and PQ a diameter of the rectangular hyperbola through the feet of the normals, then, assuming that chords of the ellipse which subtend a right angle at Q envelope another ellipse of which Q is a focus, prove geometrically (i.) that P is its second focus, (ii.) that CP, CQ are equally inclined to the major axis of the given ellipse, (iii.) that CP is to CQ in a constant ratio. E. P. ROUSE.

177. If A, B, C are three circles of which A is the inverse of B with respect to C, prove that the inverse of the circle of similitude of A and B with respect to C is the radical axis of A and B. A. LODGE.

178. If two great circles on a sphere be orthogonally projected into ellipses on a plane through the centre O, prove that the angle between the planes of the circles is equal to the angle POQ, where P and Q are the points in which tangents to the two ellipses parallel to a common diameter meet the great circle in the plane of projection. A. LODGE.

179. If a, b, c are three positive magnitudes such that $a>b>c$, $a-b>b-c$, and $\frac{1}{b}-\frac{1}{a}>\frac{1}{c}-\frac{1}{b}$, then the same inequalities will hold with respect to A, B, C, where $A\equiv m+\frac{n}{a}$, $B\equiv m+\frac{n}{b}$, $C\equiv m+\frac{n}{c}$, m and n being any positive magnitudes.

R. F. MUIRHEAD.

180. If the odds against a horse in a race be a to b, then we may call $\frac{b}{a+b}$ the apparent chance of that horse winning. Prove that, if the sum of the apparent chances of all the horses in a race be less than unity, one can arrange bets so as to make sure of winning the same sum of money whatever be the issue of the race.

R. W. GENESE.

SOLUTIONS.

Solutions of Problems in No. 8 have been received from G. HARRISON and W. P. GOUDIE. Solutions cannot however as a rule be published unless they are received in time for the next number of the *Gazette* following that in which the Problems were proposed.

Mr. Harrison's solution of Problem 94, p. 69, *To construct the bisector of the angle between two given lines which do not meet on the paper*, is as follows:

Let AB, CD be the two given lines, and let any third line cross them at E, F. Let the bisectors of the interior angles on one side of EF meet in G, and of those on the other side in H. Then GH is the line required.

Mr. Goudie's solution of the same Problem is as follows:

Take any point E on AB, and draw EF parallel to CD; bisect the angle AEF by a line meeting CD in G. Then the line bisecting EG at right angles is the one required.

PROFESSOR GENESE sends us the following note:

The force of Professor Lodge's argument on p. 68 may be emphasized as follows:—In the proof of the principle of equal fractions we reason thus,

let $$\frac{a}{b}=\frac{c}{d}=\lambda, \quad \text{......(1)}$$

then $$a=b\lambda,\ c=d\lambda,$$

$$\therefore\ (a-c)=(b-d)\lambda; \quad \text{......(2)}$$

then $\lambda=\frac{a-c}{b-d}$, unless *both* $a-c=0$ and $b-d=0$, in which case equation (2) ceases to exist as a simple equation for λ; in fact, λ=any finite quantity satisfies it, which broad statement includes the truth $\lambda=\frac{a}{b}$.

Mr. J. BRILL writes more at length mainly to the same effect.

59. *Express $yz+zx+xy$ as the sum of three positive or negative squares.*

If x, y, z are integers, there must be two among them which are both even or both odd; let y, z be the two; put $y+z=2m$, $y-z=2n$, so that $y=m+n$, $z=m-n$.

Then
$$\begin{aligned} yz+zx+xy &= (x+y)(x+z)-x^2 \\ &= (x+m+n)(x+m-n)-x^2 \\ &= (x+m)^2-x^2-n^2. \end{aligned}$$

The question was set in a St. John's College scholarship paper in 1890. It does not appear that a real answer can be found to the question if the squares are to be all positive or all negative. Another solution, given by Mr. GREENSTREET, is

$$\left\{x+\frac{\iota}{\sqrt{2}}(y-z)\right\}^2+\left\{y+\frac{\iota}{\sqrt{2}}(z-x)\right\}^2+\left\{z+\frac{\iota}{\sqrt{2}}(x-y)\right\}^2.$$

123. *If x is positive, prove that $x+x^2+\ldots+x^n$ always lies between*
$\frac{1}{2}\{(n+1)x^n+(n-1)\}$ and $\frac{1}{2}n\{(n+1)x-(n-1)\}$. (Edinburgh.)

$$\begin{aligned}
&2(x+x^2+\ldots+x^n)-\{(n+1)x^n+(n-1)\}\\
&\quad=2(x+x^2+\ldots+x^{n-1})-(n-1)(x^n+1)\\
&\quad=(x+x^{n-1})+(x^2+x^{n-2})+\ldots+(x^{n-1}+x)-(n-1)(x^n+1)\\
&\quad=\sum_{r=1}^{r=n-1}(x^r+x^{n-r}-x^n-1)=-\sum_{r=1}^{r=n-1}(1-x^r)(1-x^{n-r})<0.
\end{aligned}$$

$$\therefore\quad x+x^2+\ldots+x^n<\tfrac{1}{2}\{(n+1)x^n+(n-1)\}.$$

Again $\quad 2(x+x^2+\ldots+x^n)-n\{(n+1)x-(n-1)\}$

$$=\sum_{r=1}^{r=n-1}(n-r)(1-x)(1-x^r)>0.$$

$$\therefore\quad x+x^2+\ldots+x^n>\tfrac{1}{2}n\{(n+1)x-(n-1)\}.$$

124. *The bisector of the vertical angle of a triangle is not greater than the median from the same vertex.* (Edinburgh.)

Solution by W. E. JEFFARES.

Let ABC be the triangle, M the middle point of BC, and AX the bisector of the angle A. Produce AM to D, making MD equal to AM. Then the triangles BMD, CMA are equal in all respects (Euc. I., 4), so that $BD=AC$, and $B\hat{D}M=C\hat{A}M$.

Now of the two sides AB, AC suppose AB the greater; then $AB>BD$, $\therefore\ B\hat{D}A>B\hat{A}D$, *i.e.* $C\hat{A}M>B\hat{A}M$. Hence the line AX, which bisects the angle A, must lie within the angle CAM. Hence it can be easily proved that $A\hat{X}M>A\hat{M}X$, $\therefore\ AM>AX$.

128. *If $ABCD$ is a parallelogram, X any point on BC, and Y any point on AX, prove that the triangles DXY, YBC are equal.*

Solution by C. F. SANDBERG and W. E. JEFFARES.

Draw through Y a parallel to AD cutting AB in M and CD in N.

Then $\quad \triangle DXY=\triangle DXA-\triangle DYA$

$$=\tfrac{1}{2}\ \|\text{gram } DB-\tfrac{1}{2}\ \|\text{gram } DM$$
$$=\tfrac{1}{2}\ \|\text{gram } NB=\triangle YBC.$$

130. *A shot of mass m is discharged from a gun which together with the gun-carriage is of mass M. The gun-carriage can slide on a smooth horizontal plane. Prove that for a given charge of powder the range is a maximum for an elevation $\frac{1}{2}\cos^{-1}\dfrac{m}{2M+m}$ of the gun.* (Trin. Coll., 1890.)

Solution by R. F. MUIRHEAD.

The condition that the charge of powder is given may be taken to imply that the total kinetic energy of shot and gun is a fixed quantity, which we may denote by E. We neglect the kinetic energy of the powder after explosion, and assume that there is no elastic rebound of the gun and carriage from the horizontal plane.

Let v denote the velocity of the shot relative to the gun.
„ V „ „ „ gun and carriage. It is horizontal.
„ a „ angular elevation of the gun.
„ R „ range.

Resolving momentum horizontally, $m(v\cos a - V) = MV$. ....(1

$$\therefore\ v = (M+m)V \div m\cos a.$$

$$R = (v\cos a - V)\times \text{time of flight} = (v\cos a - V)\,.\,2v\sin a \div g$$

$$= \frac{2MVv\sin a}{mg} = \frac{2M\,.\,(M+m)V^2\sin a}{m^2 g\cos a} = V^2\tan a \times \text{constant}, \text{ ....(2)}$$

$$2E = MV^2 + m(v\cos a - V)^2 + mv^2\sin^2 a.$$

$$\therefore\ 2Em = mMV^2 + M^2V^2 + m^2v^2\sin^2 a = MV^2(M+m) + (M+m)^2V^2\tan^2 a.$$

$$\therefore\ 2Em \div (M+m) = V^2\{M + (M+m)\tan^2 a\}. \text{ ....(3)}$$

From (2) and (3) we get $R = \text{constant}\times\tan a \div \{M + (M+m)\tan^2 a\}$.

To make this a maximum $\dfrac{M}{\tan a} + (M+m)\tan a$ must be a minimum. Just as $x + \dfrac{1}{x}$ is a min. when $x = 1$, the above is a min. when $\tan^2 a = \dfrac{M}{M+m}$ or $\cos 2a = \dfrac{m}{2M+m}$.

133. *If P, P' are any two points on a central conic, and the normals at P, P' meet the major axis in G, G' respectively, prove that the projections of PG, $P'G'$ on PP' are equal.* (Cambridge.)

Solution by E. Fenwick, C. F. Sandberg, and W. E. Jeffares.

Suppose the given conic an ellipse having S, S' for foci. Then

$$PS + PS' = P'S + P'S';$$

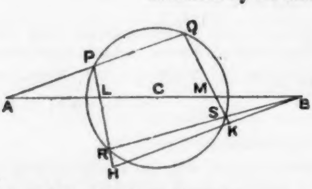
therefore $SP - SP' = S'P' - S'P$. Hence S, S' lie on *different* branches of a hyperbola having P, P' for foci. Also since PG bisects the angle SPS' internally, G lies on the directrix of the hyperbola corresponding to the focus P; similarly G' lies on the directrix corresponding to P'. Hence it follows that the projections of PG, $P'G'$ on the transverse axis of the hyperbola, viz. PP', are equal.

A similar proof holds when the given conic is a hyperbola, but it would require to be modified in the case of the parabola.

134. In this question there is a slight catch. When B projects the ball upwards, A begins to move upwards, and continues with uniform velocity; and in order that the ball may just reach him it must not, at the instant, be at rest, but have a velocity upwards equal to that of A.

135. *If two chords PQ, RS of a circle cut a given diameter in points equidistant from the centre, then so also do PR, QS and PS, QR.*

Solution by E. Budden and Dr. Mackay.



Let PQ, RS cut the diameter in A, B, so that AB is bisected at the centre C. Draw BH parallel to AP; join PR, QS cutting BH in H, K, and AB in L, M.

Then $HRS = PQS = SKB$,

$\therefore$ R, S, K, H are concyclic points.

$\therefore$ $BK\,.\,BH = BS\,.\,BR = AP\,.\,AQ$,

since $CA = CB$.

Again by similar triangles,

$$\frac{AP}{AL} = \frac{BH}{BL} \text{ and } \frac{AQ}{AM} = \frac{BK}{BM}, \quad \therefore\ \frac{AP\,.\,AQ}{AL\,.\,AM} = \frac{BH\,.\,BK}{BL\,.\,BM};$$

$$\therefore\ AL \cdot AM = BL \cdot BM.$$

Hence $$\frac{AL}{BM} = \frac{BL}{AM} = \frac{AL+BL}{BM+AM} = 1;$$

$$\therefore\ AL = BM, \text{ and } LM \text{ is bisected at } C.$$

For three other solutions of this question, two of which are by elementary geometry, see Dr. Mackay's paper in the *Proceedings of the Edinburgh Mathematical Society*, vol. iii., pp. 38-42.

136. *The inverse of a circle with respect to a point outside its plane is a circle.*

It is easily proved that the inverse of a sphere with respect to any point is a sphere. Hence a circle, which is the intersection of two spheres, inverts into a curve which is the intersection of two spheres, *i.e.* it inverts into a circle.

137. *In an acute-angled triangle R^2 lies between $\frac{1}{8}(a^2+b^2+c^2)$ and $\frac{1}{9}(a^2+b^2+c^2)$, and in an obtuse-angled triangle R^2 is greater than $\frac{1}{8}(a^2+b^2+c^2)$.*

Solution by W. E. Jeffares.

$$\begin{aligned} a^2+b^2+c^2 &= 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \\ &= 8R^2(1+\cos A \cdot \cos B \cdot \cos C). \end{aligned}$$

When the triangle is acute-angled $\cos A \cdot \cos B \cdot \cos C$ is positive, and is not greater than $\frac{1}{8}$. Hence $a^2+b^2+c^2$ lies between $8R^2$ and $9R^2$.

When the triangle is obtuse-angled $\cos A \cdot \cos B \cdot \cos C$ is negative, and $a^2+b^2+c^2$ is less than $8R^2$.

138. *Find the five linear factors of $x^5+y^5+z^5-5xyz(x^2-yz)$.*

If we put $x=-(y+z)$, the expression becomes

$$-(y+z)^5+y^5+z^5+5yz(y+z)(y^2+yz+z^2)=0.$$

Again, if we change y, z to ωy, $\frac{1}{\omega}z$, where ω is any fifth root of unity, the given expression does not alter; hence $x+\omega y+\frac{1}{\omega}z$ is a factor, and by giving to ω its five different values we have the five linear factors.

139. *Find the conditions that the cubic $ax^3+3bx^2+3cx+d=0$ should have three real roots between ± 1.*

There is first the known condition that the roots of the cubic should be all real, viz.

$$4(b^2-ac)(c^2-bd) > (bc-ad)^2.$$

The remaining conditions may be found by putting $z=\frac{x-1}{x+1}$, or $x=\frac{1+z}{1-z}$. Then all the three values of z, corresponding to the three values of x, will be negative; and conversely, if z has any negative value, the corresponding value of x lies between ± 1. Hence the required conditions are that the cubic for z, viz.

$$a(1+z)^3+3b(1+z)^2(1-z)+3c(1+z)(1-z)^2+d(1-z)^3=0,$$

or $(a-3b+3c-d)z^3+3(a-b-c+d)z^2+3(a+b-c-d)z+(a+3b+3c+d)=0$,

should have all its coefficients of the same sign. These conditions may be expressed thus

(i.) $a+3c$ and $a-c$ must be of the same sign,

(ii.) $a-c$ must be numerically greater than $b-d$,

(iii.) $a+3c$ „ „ „ $3b+d$,

(iv.) $4(b^2-ac)(c^2-bd) > (bc-ad)^2$.

140. *If m, n, x are positive integers, of which m, n are prime to one another, prove that*

$$1+x+\ldots+x^{m-1} \quad \textit{and} \quad 1+x+\ldots+x^{n-1}$$

are arithmetically prime to one another.

Since m, n are prime to one another, two positive integers p, q can be found such that $pm-qn=1$.

Now x^m-1 divides $x^{pm}-1$, and x^n-1 divides $x^{qn}-1$; hence the arithmetic H.C.F. of x^m-1 and x^n-1 divides

$$(x^{pm}-1)-(x^{qn}-1)=x^{pm}-x^{qn}=x^{qn}(x-1).$$

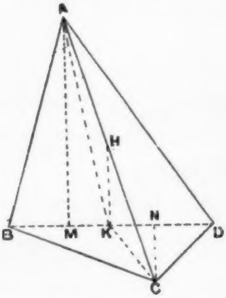
The numbers x^m-1, x^n-1 can therefore have no other arithmetic common factor than $x-1$, *i.e.* the numbers $x^{m-1}+\ldots+1$, $x^{n-1}+\ldots+1$ are arithmetically prime to one another.

Or thus: The two given numbers, if expressed in the scale of x, will have m and n digits respectively, all equal to unity. Hence if their H.C.F. be worked out by the ordinary process, every remainder will have all its digits equal to unity, so that the H.C.F. must be unity.

141. *Prove that if the areas of the four faces of a tetrahedron are equal, each edge is equal to the opposite edge.*

Solution by J. C. M. Garnett.

Let $ABCD$ be the tetrahedron; draw AM, CN perpendicular to BD; bisect MN at K, and AC at H.



Then if AHC be projected on the plane BCD into $A'H'C$, H' will be the middle point of $A'C$; and, since MA', CN are both perpendicular to BD, so also is KH', and therefore also KH.

Now since the triangles ABD, CBD are equal, their heights AM, CN are equal; hence in the triangles AMK, CNK we have $KA=KC$; and from the triangles KHA, KHC, KH is perpendicular to AC.

Hence HK is the common perpendicular to AC and BD; and since it bisects AC, it must also by symmetry bisect BD.

Hence $BM=ND$; and in the triangles AMB, CND, $AB=CD$. Therefore each edge of the tetrahedron is equal to the opposite edge.

143. *Prove that the result of eliminating the constants in the general equation of a conic by differentiation is*

$$\frac{d^3}{dx^3}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\}=0. \qquad \text{(Oxford.)}$$

Let the equation to the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Solving for y, we have

$$y=Dx+E\pm(Ax^2+2Bx+C)^{\frac{1}{2}},$$

where A, B, C, D, E are constants.

Differentiating, $$\frac{dy}{dx}=D\pm\frac{Ax+B}{(Ax^2+2Bx+C)^{\frac{1}{2}}};$$

and
$$\frac{d^2y}{dx^2} = \pm\frac{A(Ax^2+2Bx+C)-(Ax+B)^2}{(Ax^2+2Bx+C)^{\frac{3}{2}}}$$
$$= \pm\frac{AC-B^2}{(Ax^2+2Bx+C)^{\frac{3}{2}}};$$
$$\therefore\ \left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = (AC-B^2)^{-\frac{2}{3}}(Ax^2+2Bx+C);$$
$$\therefore\ \frac{d^3}{dx^3}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\} = 0.$$

[Solutions of Problems 145 to 149 are held over for the next number of the *Gazette.*]

REVIEWS AND NOTICES.

Advanced Mechanics. Vol. II.—Statics. By WILLIAM BRIGGS, M.A., and G. H. BRYAN, Sc.D., F.R.S. ("The Organized Science Series," London, W. B. Clive, 288 pp., price 3/6.) This book adds one more to the fairly long list of elementary works which are designed to help beginners in Dynamical Science. It falls into line with modern usage in basing the proof of the parallelogram of forces on Newton's kinetical definition of force; so that we may now congratulate ourselves on the disappearance of such "statical" methods as that of Duchayla. The present work is throughout extremely systematic, and it abounds in examples, examination papers, etc., all in their proper places as applications and illustrations of the text.

The principle of Virtual Work is introduced at an early stage and well explained; but (p. 41) in the simple case in which a particle is sustained by a horizontal force on a smooth inclined plane, while the value of this force in terms of the weight is easily deduced by giving (*actually giving*) the particle a displacement up the plane, we do not see why the authors take the trouble of specially saying that "it would be less easy to determine the Reaction by means of the Principle of Work." On the contrary, it is quite as easy to determine the Reaction as it is to determine the horizontal force by this principle; for we have merely to imagine the particle to receive a vertical displacement instead of one up the plane, and the Reaction is at once determined. The authors may, perhaps, justify themselves on the ground that they make a distinction between the "Principle of Work," which is introduced at p. 34, and the "Principles of Virtual Work and Virtual Velocities," which are not introduced until p. 216 is reached. The distinction, however, is not a valid one, because their "Principle of Work" is this: "If a particle, acted upon by any number of forces *in equilibrium, is moved* from one position to another, the algebraic sum of the works done by the several forces is zero" (p. 34. The italics are ours.) A beginner may justly ask "How is the particle, which is in equilibrium, *moved*? It cannot be by the forces themselves: it must be, therefore, by some other agent, and hence the motion assumed can be only a virtual one." Moreover, if actual motion is contemplated, it must be one entirely unaccompanied by acceleration—which fact is certainly not properly explained in the above enunciation, and could not be understood by the student at such an early stage. It ought to be clearly understood that when no accelerations or gain of kinetic energy are involved, there is no distinction whatever between the "Principle of Work" and the "Principle of Virtual Work." The manner in which the authors introduce this latter is somewhat amusing. It is this:

"VIRTUAL WORK.—We often find it convenient for the sake of argument to *suppose* a particle displaced from one position A to another B, although the particle may have no tendency to move in this direction, or, perhaps, in any direction," (p. 216). Are there any other physical principles which exist "for the sake of argument?"

The authors have, we think, done well to treat of the equilibrium of forces acting on a particle before discussing the conditions of equilibrium of an

extended rigid body; but we would point out that the crane (p. 45) is not properly a case of particle equilibrium, because the discussion assumes the forces at the extremities of the jib, to be directed along the jib, and this assumption ought to be justified by a consideration of the separate equilibrium of the jib itself.

The resultant of two parallel forces is deduced (p. 60) by a neat (and to us novel) method, viz., from the case of three forces acting at the middle points of the sides of any triangle, perpendicularly and proportional to those sides.

Moments are discussed and well illustrated in Chap. V. We have heard of the work of a couple; but a sudden and uncontrollable merriment was excited when we saw that Art. 73 was introduced to us in large type as the "WORK OF A MOMENT!"

We are glad to note that in the discussion of machines, which occupies a large part of the volume, the authors adopt the terms *effort* and *resistance* instead of the execrable "power" and "weight" of the old books, and alas, also of some of the new books and even modern examination papers!

One good feature of the book consists in the many *numerical* examples given—a kind of example hateful to the ordinary university mathematician. The figures, and especially those of machines, are throughout very good, and indicate much care and trouble on the part of the authors; but may we inquire the nature of the cargo conveyed by the cart on p. 163. Is it a load of glass wool? In conclusion, the work will be found extremely helpful to beginners on account of its clearness and thoroughness; and it will be no less useful to teachers on account of its abundance of excellent examples. G.M.M.

Elementary Solid Geometry and Mensuration. By PROFESSOR HENRY DALLAS THOMPSON, D.Sc., Ph.D. (Macmillan, 1896. Price 6s.) This is an excellent carefully worded text book, the early part of which follows the same lines as Mr. R. B. Hayward's book on Solid Geometry, so much so in fact as to cause surprise that no reference is made to it in the Preface. The diagrams are, as in all American books, exceedingly clear. It is a question whether a student could profitably spare the time to learn all the propositions as thoroughly as he usually learns the plane Geometry of Euclid, but that probably is not intended. In some details the book differs from Mr. Hayward's, specially in the definition of a normal to a plane; but the main lines of the two books are the same, and the arguments in many instances practically identical; though occasionally the explanations are somewhat more amplified in the American books, and the figures are better. More ground also is covered in the latter chapters. Intersections of lines and planes are first dealt with, then angles of lines and planes, dihedral and polyhedral angles. Then polyhedra, the five regular polyhedra, and cylinders and cones are dealt with. Next follows an excellent chapter on the sphere and spherical triangles, and a short but clearly expressed chapter on the plane sections of a circular cone. It would have been better, perhaps, if the theorem that tangents from a given point to a given sphere are all equal, which is used and proved in this chapter, had been given as a substantive proposition in the preceding chapter to which it belongs. The last chapter (VIII) is devoted to the mensuration of some of the simple solids, with a special dissertation on the Prismatoid, in which, what is generally known as Simpson's Rule or the Prismoidal formula is introduced. The author attributes it to Steiner, a German mathematician, in 1842; but Simpson made use of it nearly 100 years earlier, and we believe it can be traced back to Newton. It is by far the most important formula for the volumes of solids, and, indeed, is almost the only necessary formula, the volumes of all the simple solids and their frusta being derivable from it. A.L.

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